

## Three-body interactions in Fermi systems

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We show that the contributions of three-quasiparticle interactions to normal Fermi systems at low energies and temperatures are suppressed by  $n_q/n$  compared to two-body interactions, where  $n_q$  is the density of excited or added quasiparticles and  $n$  is the ground-state density. For finite Fermi systems, three-quasiparticle contributions are suppressed by the corresponding ratio of particle numbers  $N_q/N$ . This is illustrated for polarons in strongly interacting spin-polarized Fermi gases and for valence neutrons in neutron-rich calcium isotopes.

### 1. Introduction

Three-nucleon (3N) forces and advancing microscopic many-body methods are a frontier in the physics of nuclei and nucleonic matter in stars. New facets of 3N forces are revealed in neutron-rich nuclei, such as their role in determining the location of the neutron dripline [1, 2] and in elucidating the doubly-magic nature of  $^{48}\text{Ca}$  [3]. Three- and higher-body forces are also the dominant uncertainty in constraining the properties of neutron-rich matter at nuclear densities and thus the structure of neutron stars [4, 5]. At the same time, 3N forces are at the center of developing systematic interactions based on effective field theory (EFT) [6], thus linking the nuclear forces

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frontier with the experimental exploration of neutron-rich nuclei.

For light nuclei, ab-initio methods have unambiguously established the quantitative role of 3N forces for ground-state properties, excitations and reactions [7, 8]. These results are consistent with estimates of 3N force contributions  $\langle V_{3N} \rangle \sim 0.1 - 1$  MeV per triplet in few-nucleon systems [9]. However, this scaling with the number of triplets, which would suggest that 3N forces become more important than two-nucleon (NN) interactions with increasing particle number (independent of the density), has to break down for larger nuclei. This is because nuclear forces have a finite range  $R$  (pion-exchange or shorter), so that in larger systems the interactions of three particles are restricted to volumes  $R^3 < L^3$ , where  $L$  is the size of the nucleus. As a result, in nuclear matter 3N forces scale with the density  $n$  (and not the number of triplets to pairs) compared to two-nucleon interactions  $\langle V_{3N} \rangle \sim n R^3 \langle V_{NN} \rangle$  (compare, e.g., the results in Ref. [10]).

For larger nuclei, three-body interactions can be classified according to a finite-density reference state. In Fermi liquid theory, this corresponds to the interacting ground state, which is often taken to be a core nucleus. While 3N force contributions to the energy of the core nucleus are important, at the level of accuracy of present calculations for medium-mass nuclei, there is no evidence for residual three-body interactions between *valence* nucleons. For example, a recent analysis concludes [11]: “So far, no evidence was found for the effects of three-body interactions on states of valence nucleons. In case where rather pure shell-model configurations were observed, states and energies were well determined by effective two-body interactions”. Clearly this must depend on the number of particles in the core, because in light nuclei with only valence particles, three-body interactions are significant. In the framework of Fermi liquid theory, this suggests that three-quasiparticle interactions are small at low energies, both for excitations and when valence nucleons are added.

In this paper, we discuss the impact of three-quasiparticle interactions in normal Fermi systems. These questions have been touched on briefly in the literature [12, 13]. For example, Brandow writes: “The weakness of the apparent three-body effects is the essential content of the statement that a Fermi liquid may be viewed as a low-density gas of weakly interacting quasiparticles”. Considering the developments for 3N forces, it is however important to revisit these issues. After an introduction to Fermi liquid theory, we show in Section 3 that the contributions of three-quasiparticle interactions to normal Fermi systems at low energies and temperatures are suppressed by  $n_q/n$  compared to two-body interactions, where  $n_q$  is

the density of excited or added quasiparticles and  $n$  is the ground-state density, or by the corresponding ratio of particle numbers  $N_q/N$ . This explains why the shell model with effective two-body interactions works so well. It also enables us to estimate at which level residual three-body interactions are expected to contribute. These results demonstrate that there is a change from few-body systems and light nuclei to normal Fermi systems. For larger nuclei and nucleonic matter, the contributions from residual three-body interactions are small when the system is weakly excited (including excitations where valence nucleons are added), even if 3N forces are significant for the interacting ground state (the core nucleus).

We dedicate this paper to Gerry Brown on the occasion of his 85th birthday. Gerry is one of the pioneers in the theory and microscopic understanding of Fermi systems. We were extremely fortunate to start on many-body problems and Fermi liquid theory with him as a teacher and mentor when we were students in Stony Brook. Using Gerry's words in "Fly with eagles" [14], he has been an eagle for generations of nuclear theorists and certainly for both of us.

## 2. Fermi liquid theory as an effective theory

Much of our understanding of strongly interacting Fermi systems at low energies and temperatures goes back to the seminal work of Landau in the late fifties [15–17]. Landau was able to express macroscopic observables in terms of microscopic properties of the elementary excitations, the so-called quasiparticles, and their residual interactions. In order to illustrate Landau's arguments here, we consider a uniform system of non-relativistic spin-1/2 fermions at zero temperature.

Landau assumed that the low-energy, elementary excitations of the interacting system can be described by effective degrees of freedom, the quasiparticles. Due to translational invariance, the states of the uniform system are eigenstates of the momentum operator. The quasiparticles are much like single-particle states in the sense that for each momentum there is a well-defined quasiparticle energy.<sup>a</sup> Landau assumed that there is a one-to-one correspondence between the quasiparticles and the single-particle states

<sup>a</sup>In general, a quasiparticle state is not an energy eigenstate, but rather a resonance with a non-zero width. For quasiparticles close to the Fermi surface, the width is small and the corresponding life-time is large; hence the quasiparticle concept is useful for time scales short compared to the quasiparticle life-time.

of a free Fermi gas. For a superfluid system, this one-to-one correspondence does not exist, and Landau's theory must be suitably modified, as discussed by Larkin and Migdal [18] and Leggett [19].

The one-to-one correspondence starts from a free Fermi gas consisting of  $N$  particles, where the ground state is given by a filled Fermi sphere in momentum space. The particle number density  $n$  and the ground-state energy  $E_0$  are given by (with  $\hbar = c = 1$ )

$$n = \frac{1}{V} \sum_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma}^0 = \frac{k_F^3}{3\pi^2} \quad \text{and} \quad E_0 = \sum_{\mathbf{p}\sigma} \frac{\mathbf{p}^2}{2m} n_{\mathbf{p}\sigma}^0 = \frac{3}{5} \frac{k_F^2}{2m} N, \quad (1)$$

where  $k_F$  denotes the Fermi momentum,  $V$  the volume, and  $n_{\mathbf{p}\sigma}^0 = \theta(k_F - |\mathbf{p}|)$  is the Fermi-Dirac distribution function at zero temperature for particles with momentum  $\mathbf{p}$ , spin projection  $\sigma$ , and mass  $m$ . By adding particles or holes, the distribution function is changed by  $\delta n_{\mathbf{p}\sigma} = n_{\mathbf{p}\sigma} - n_{\mathbf{p}\sigma}^0$ , and the total energy of the system by

$$\delta E = E - E_0 = \sum_{\mathbf{p}\sigma} \frac{\mathbf{p}^2}{2m} \delta n_{\mathbf{p}\sigma}. \quad (2)$$

When a particle is added in the state  $\mathbf{p}\sigma$ , one has  $\delta n_{\mathbf{p}\sigma} = 1$  and when a particle is removed (a hole is added)  $\delta n_{\mathbf{p}\sigma} = -1$ .

In the interacting system the corresponding state is one with a quasiparticle added or removed, and the change in energy is given by

$$\delta E = \sum_{\mathbf{p}\sigma} \varepsilon_{\mathbf{p}\sigma} \delta n_{\mathbf{p}\sigma}, \quad (3)$$

where  $\varepsilon_{\mathbf{p}\sigma} = \delta E / \delta n_{\mathbf{p}\sigma}$  denotes the quasiparticle energy. When two or more quasiparticles are added to the system, an additional term takes into account the interaction between the quasiparticles:

$$\delta E = \sum_{\mathbf{p}\sigma} \varepsilon_{\mathbf{p}\sigma}^0 \delta n_{\mathbf{p}\sigma} + \frac{1}{2V} \sum_{\mathbf{p}_1\sigma_1, \mathbf{p}_2\sigma_2} f_{\mathbf{p}_1\sigma_1 \mathbf{p}_2\sigma_2} \delta n_{\mathbf{p}_1\sigma_1} \delta n_{\mathbf{p}_2\sigma_2}. \quad (4)$$

Here  $\varepsilon_{\mathbf{p}\sigma}^0$  is the quasiparticle energy in the ground state. In Section 3, we will show that the expansion in  $\delta n$  is general and does not require weak interactions. The small expansion parameter in Fermi liquid theory is the density of quasiparticles, or equivalently the excitation energy, and not the strength of the interaction. This allows a systematic treatment of strongly interacting systems at low temperatures.

In normal Fermi systems, the quasiparticle concept makes sense only for states close to the Fermi surface, where the quasiparticle life-time  $\tau_{\mathbf{p}} \sim$

$(p - k_F)^2$  is long [20]. In particular, states deep in the Fermi sea, which are occupied in the ground-state distribution, do not correspond to well-defined quasiparticles. Accordingly, we refer to the interacting ground state that corresponds to a filled Fermi sea in the non-interacting system as a state with no quasiparticles. In a weakly excited state the quasiparticle distribution  $\delta n_{\mathbf{p}\sigma}$  is generally non-zero only for states close to the Fermi surface.

The second term in Eq. (4), the quasiparticle interaction  $f_{\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2}$ , has no correspondence in the non-interacting Fermi gas. In an excited state with more than one quasiparticle, the quasiparticle energy is modified according to

$$\varepsilon_{\mathbf{p}\sigma} = \frac{\delta E}{\delta n_{\mathbf{p}\sigma}} = \varepsilon_{\mathbf{p}\sigma}^0 + \frac{1}{V} \sum_{\mathbf{p}_2\sigma_2} f_{\mathbf{p}\sigma\mathbf{p}_2\sigma_2} \delta n_{\mathbf{p}_2\sigma_2}, \quad (5)$$

where the changes are effectively proportional to the quasiparticle density.

The quasiparticle interaction can be understood microscopically from the second variation of the energy with respect to the quasiparticle distribution,

$$f_{\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2} = V \frac{\delta^2 E}{\delta n_{\mathbf{p}_1\sigma_1} \delta n_{\mathbf{p}_2\sigma_2}} = V \frac{\delta \varepsilon_{\mathbf{p}_1\sigma_1}}{\delta n_{\mathbf{p}_2\sigma_2}}. \quad (6)$$

Diagrammatically, this variation corresponds to cutting one of the fermion lines in a given energy diagram and labeling the incoming and outgoing fermion by  $\mathbf{p}_1\sigma_1$ , followed by a second variation leading to  $\mathbf{p}_2\sigma_2$ . For the uniform system, the resulting contributions to  $f_{\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2}$  are quasiparticle reducible in the particle-particle and in the exchange particle-hole (induced interaction) channels, but irreducible in the direct particle-hole (zero sound) channel [21–23]. The zero-sound-channel reducible diagrams are generated by the particle-hole scattering equation [17]. With Babu, one of Gerry's seminal contributions to Fermi liquid theory was to derive an integral equation that self-consistently takes into account induced interactions due to the polarization of the medium [23]. The Babu-Brown induced interaction is still one of the few non-perturbative approaches for calculating Fermi liquid parameters that have been implemented in practice.

Landau's theory of normal Fermi liquids is an effective low-energy theory in the modern sense [24, 25]. The effective theory incorporates the symmetries of the system and the low-energy couplings can be fixed by experiment or calculated microscopically based on the underlying theory. In an isotropic and spin-saturated system, such as liquid  $^3\text{He}$ , the quasiparticle

interaction can be decomposed as

$$f_{\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2} = f_{\mathbf{p}_1\mathbf{p}_2}^s + f_{\mathbf{p}_1\mathbf{p}_2}^a \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2, \quad (7)$$

which is the most general form consistent with  $SU(2)$  spin symmetry.<sup>b</sup> For nuclear systems, the quasiparticle interaction includes additional terms that take into account the isospin dependence and non-central tensor contributions [26–28]. However, for our discussion here, the spin and isospin dependence is not important.

For the uniform system, Eq. (6) yields the quasiparticle interaction only for forward scattering (low momentum transfers). In the particle-hole channel, this corresponds to the long-wavelength limit. This restriction, which is consistent with considering low excitation energies, constrains the momenta  $\mathbf{p}_1$  and  $\mathbf{p}_2$  to be close to the Fermi surface,  $|\mathbf{p}_1| = |\mathbf{p}_2| = k_F$ . The quasiparticle interaction then depends only on the angle between  $\mathbf{p}_1$  and  $\mathbf{p}_2$ . It is convenient to expand this dependence on Legendre polynomials

$$f_{\mathbf{p}_1\mathbf{p}_2}^{s/a} = f^{s/a}(\cos\theta_{\mathbf{p}_1\mathbf{p}_2}) = \sum_l f_l^{s/a} P_l(\cos\theta_{\mathbf{p}_1\mathbf{p}_2}), \quad (8)$$

and to define the dimensionless Landau Parameters  $F_l^{s/a}$  by

$$F_l^{s/a} = N(0) f_l^{s/a}, \quad (9)$$

where  $N(0) = \frac{1}{V} \sum_{\mathbf{p}\sigma} \delta(\varepsilon_{\mathbf{p}\sigma} - \mu) = m^* k_F / \pi^2$  denotes the quasiparticle density of states at the Fermi surface.

The Landau parameters can be directly related to macroscopic properties of the system. Here we mention only the specific heat  $c_V$ , which at low temperature is determined by the effective mass given by  $F_1^s$ ,

$$\frac{m^*}{m} = 1 + \frac{F_1^s}{3}, \quad (10)$$

$$c_V = \frac{m^* k_F}{3} k_B^2 T, \quad (11)$$

and the incompressibility  $\kappa$ , which is related to  $F_0^s$ ,

$$\kappa = -\frac{9V}{n} \frac{\partial P}{\partial V} = \frac{3k_F^2}{m^*} (1 + F_0^s). \quad (12)$$

Fermi liquid theory has been very successful in describing low-temperature Fermi liquids, in particular liquid  $^3\text{He}$  [20]. The first applications to nuclear systems were pioneered by Migdal [26] and first microscopic calculations for nuclei and nuclear matter by Gerry and collaborators (for a review, see

<sup>b</sup>In nuclear physics the notation  $f_{\mathbf{p}_1\mathbf{p}_2} = f_{\mathbf{p}_1\mathbf{p}_2}^s$  and  $g_{\mathbf{p}_1\mathbf{p}_2} = f_{\mathbf{p}_1\mathbf{p}_2}^a$  is generally used.

Ref. [27]). Recently, advances using renormalization group (RG) methods for nuclear forces [29] and Fermi systems [24] have lead to the development of a non-perturbative RG approach for nucleonic matter [30], to a first complete study of the spin structure of induced interactions [28], and to new calculations of Fermi liquid parameters [31, 32].

### 3. Three-quasiparticle interactions

In Section 2, we introduced Fermi liquid theory as an expansion in the density of quasiparticles  $\delta n/V$ . In applications of Fermi liquid theory to date, even for liquid  $^3\text{He}$ , which is a very dense and strongly interacting system, this expansion is truncated after the second-order  $(\delta n)^2$  term, including only pairwise interactions of quasiparticles. However, for a strongly interacting system, there is a priori no reason that three-body (or higher-body) interactions between quasiparticles are small. In this section, we discuss the convergence of this expansion. Three-quasiparticle interactions arise from iterated two-body forces, leading to three- and higher-body clusters in the linked-cluster expansion, or through many-body forces. While three-body forces play an important role in nuclear physics [6], little is known about them in other Fermi liquids. Nevertheless, in strongly interacting systems, the contributions of many-body clusters can in general be significant, leading to potentially important  $(\delta n)^3$  terms in the Fermi liquid expansion, also in the absence of three-body forces:

$$\delta E = \sum_1 \varepsilon_1^0 \delta n_1 + \frac{1}{2V} \sum_{1,2} f_{1,2}^{(2)} \delta n_1 \delta n_2 + \frac{1}{6V^2} \sum_{1,2,3} f_{1,2,3}^{(3)} \delta n_1 \delta n_2 \delta n_3. \quad (13)$$

Here  $f_{1,\dots,n}^{(n)}$  denotes the  $n$ -quasiparticle interaction (the Landau interaction is  $f \equiv f^{(2)}$ ) and we have introduced the short-hand notation  $n \equiv \mathbf{p}_n, \sigma_n$ .

In order to better understand the expansion, Eq. (13), around the interacting ground state with  $N$  fermions, consider exciting or adding  $N_q$  quasiparticles with  $N_q \ll N$ . The microscopic contributions from many-body clusters or from many-body forces can be grouped into diagrams containing zero, one, two, three, or more quasiparticle lines. The terms with zero quasiparticle lines contribute to the interacting ground state for  $\delta n = 0$ , whereas the terms with one, two, and three quasiparticle lines contribute to  $\varepsilon_1^0$ ,  $f_{1,2}^{(2)}$ , and  $f_{1,2,3}^{(3)}$ , respectively (these also depend on the ground-state density due to the  $N$  fermion lines). The terms with more than three quasiparticle lines would contribute to higher-quasiparticle interactions. Because a quasiparticle line replaces a line with  $N$  fermions when going from  $\varepsilon_1^0$  to

$f_{1,2}^{(2)}$ , and from  $f_{1,2}^{(2)}$  to  $f_{1,2,3}^{(3)}$ , it is intuitively clear that the contributions due to three-quasiparticle interactions are suppressed by  $N_q/N$  compared to two-quasiparticle interactions, and that the Fermi liquid expansion is effectively an expansion in  $N_q/N$  or  $n_q/n$  [33]. This will be discussed in detail and illustrated with examples in the following sections.

### 3.1. General considerations

Fermi liquid theory applies to normal Fermi systems at low energies and temperatures, or equivalently at low quasiparticle densities. We first consider excitations that conserve the net number of quasiparticles,  $\delta N = \sum_{\mathbf{p}\sigma} \delta n_{\mathbf{p}\sigma} = 0$ , so that the number of quasiparticles equals the number of quasiholes. This corresponds to the lowest energy excitations of normal Fermi liquids. We denote their energy scale by  $\Delta$ . Excitations with one valence particle or quasiparticle added start from energies of order the chemical potential  $\mu$ . In the case of  $\delta N = 0$ , the contributions of two-quasiparticle interactions are of the same order as the first-order  $\delta n$  term, but three-quasiparticle interactions are suppressed by  $\Delta/\mu$  [13]. This is the reason that Fermi liquid theory with only two-body Landau parameters is so successful in describing even strongly interacting and dense Fermi liquids. This counting is best seen from the variation of the free energy  $F = E - \mu N$ ,

$$\begin{aligned} \delta F &= \delta(E - \mu N) \\ &= \sum_1 (\varepsilon_1^0 - \mu) \delta n_1 + \frac{1}{2V} \sum_{1,2} f_{1,2}^{(2)} \delta n_1 \delta n_2 + \frac{1}{6V^2} \sum_{1,2,3} f_{1,2,3}^{(3)} \delta n_1 \delta n_2 \delta n_3, \end{aligned} \quad (14)$$

which for  $\delta N = 0$  is equivalent to  $\delta E$  of Eq. (13). The quasiparticle distribution is  $|\delta n_{\mathbf{p}\sigma}| \sim 1$  within a shell around the Fermi surface  $|\varepsilon_{\mathbf{p}\sigma}^0 - \mu| \sim \Delta$ . The first-order  $\delta n$  term is therefore proportional to  $\Delta$  times the number of quasiparticles  $\sum_{\mathbf{p}\sigma} |\delta n_{\mathbf{p}\sigma}| = N_q \sim N(\Delta/\mu)$ ,

$$\sum_1 (\varepsilon_1^0 - \mu) \delta n_1 \sim \frac{N\Delta^2}{\mu}. \quad (15)$$

Correspondingly, the contribution of two-quasiparticle interactions yields

$$\frac{1}{2V} \sum_{1,2} f_{1,2}^{(2)} \delta n_1 \delta n_2 \sim \frac{1}{V} \langle f^{(2)} \rangle \left( \frac{N\Delta}{\mu} \right)^2 \sim \langle F^{(2)} \rangle \frac{N\Delta^2}{\mu}, \quad (16)$$



where  $\langle F^{(2)} \rangle = n \langle f^{(2)} \rangle / \mu$  is an average dimensionless coupling on the order of the Landau parameters. Even in the strongly interacting, scale-invariant case (see Section 3.2)  $\langle f^{(2)} \rangle \sim 1/k_F$ ; hence  $\langle F^{(2)} \rangle \sim 1$  and the contribution of two-quasiparticle interactions is of the same order as the first-order term. However, the three-quasiparticle contribution is of order

$$\frac{1}{6V^2} \sum_{1,2,3} f_{1,2,3}^{(3)} \delta n_1 \delta n_2 \delta n_3 \sim \frac{n^2}{\mu} \langle f^{(3)} \rangle \frac{N\Delta^3}{\mu^2} \sim \langle F^{(3)} \rangle \frac{N\Delta^3}{\mu^2}. \quad (17)$$

Therefore at low excitation energies this is suppressed by  $\Delta/\mu$ , compared to two-quasiparticle interactions, even if the dimensionless three-quasiparticle interaction  $\langle F^{(3)} \rangle = n^2 \langle f^{(3)} \rangle / \mu$  is strong (of order 1). Similarly, higher  $n$ -body interactions are suppressed by  $(\Delta/\mu)^{n-2}$ . Normal Fermi systems at low energies are weakly coupled in this sense. The small parameter is the ratio of the excitation energy per particle to the chemical potential. These considerations hold for *all* normal Fermi systems where the underlying interparticle interactions are finite range.

The Fermi liquid expansion in  $\Delta/\mu$  is equivalent to an expansion in  $N_q/N \sim \Delta/\mu$ , the ratio of the number of quasiparticles and quasiholes  $N_q$  to the number of particles  $N$  in the interacting ground state, or an expansion in the density of excited quasiparticles over the ground-state density,  $n_q/n$ .

We conclude this section with a discussion of the expansion for the energy  $\delta E$  given by Eq. (13), for the case where  $N_q$  quasiparticles or valence particles are added to a Fermi-liquid ground state. In this case,  $\delta N \neq 0$  and the first-order term is

$$\sum_1 \varepsilon_1^0 \delta n_1 \sim \mu N_q \sim \mu \frac{N\Delta}{\mu} \sim N\Delta, \quad (18)$$

while the contribution of two-quasiparticle interactions is suppressed by  $N_q/N \sim \Delta/\mu$  and that of three-quasiparticle interactions by  $(N_q/N)^2$ . Therefore, either for  $\delta N = 0$  or  $\delta N \neq 0$ , the contributions of three-quasiparticle interactions are suppressed for normal Fermi systems at low excitation energies. In the following sections, we will illustrate this for polarons in strongly interacting spin-polarized Fermi gases and for valence neutrons in neutron-rich calcium isotopes.

### 3.2. Strongly interacting spin-polarized Fermi gases

Experiments with spin-polarized Fermi gases [34–38] enable a unique exploration of strongly interacting Fermi systems and universal properties. We consider a system with two spin states and large S-wave scattering length

interactions. In the limit of extreme population imbalance, the physics is governed by a single spin-down fermion interacting strongly with the spin-up Fermi sea. This spin-down fermion forms a quasiparticle, the so-called Fermi polaron [39], with energy  $E_p$  and effective mass  $m_p^*$ . Polarons have been directly observed in cold atomic gases using rf spectroscopy [37].

For large scattering lengths,  $1/a_s = 0$ , the polaron energy is universal (it depends only on the density of spin-up fermions):  $E_p = \eta \varepsilon_{F\uparrow}$ , where  $\varepsilon_{F\uparrow} = (6\pi^2 n_\uparrow)^{2/3}/(2m)$  is the spin-up Fermi energy [40]. The polaron binding  $\eta = -0.615$  and effective mass  $m_p^*/m = 1.20(2)$  have been determined using Monte-Carlo methods [41] and are in excellent agreement with experiment [38]. The polaron energy constrains the energy gain for large asymmetries in the Fermi liquid phase of spin-polarized Fermi gases. This determines the critical polarization for superfluidity and sets limits on the phase diagram and the existence of partially-polarized phases [36, 40, 42].

We can expand the strongly interacting spin-polarized Fermi gas around the fully polarized system with  $N_\uparrow$  particles by adding  $N_q = N_\downarrow \ll N_\uparrow$  polarons. Following Eq. (13), the change in the energy density is given by [43]

$$\frac{\delta E_\downarrow}{V} = \varepsilon_1^0 n_\downarrow + \frac{1}{2} f^{(2)} n_\downarrow^2 + \frac{1}{6} f^{(3)} n_\downarrow^3, \quad (19)$$

where  $\varepsilon_1^0$  is the average quasiparticle energy with contributions from both the polaron binding and the kinetic energy (with the single-polaron effective mass),

$$\varepsilon_1^0 = \eta \varepsilon_{F\uparrow} + \frac{3}{5} \frac{(6\pi^2 n_\downarrow)^{2/3}}{2m_p^*}. \quad (20)$$

The average two-quasiparticle interaction  $f^{(2)}$  is due to induced interactions mediated by the spin-up Fermi sea [44] and scales with the Fermi energy and the density of spin-up fermions,

$$f^{(2)} = \frac{\varepsilon_{F\uparrow}}{n_\uparrow} F^{(2)}, \quad (21)$$

and correspondingly for the three-quasiparticle interaction,

$$f^{(3)} = \frac{\varepsilon_{F\uparrow}}{n_\uparrow^2} F^{(3)}. \quad (22)$$

For large scattering lengths,  $1/a_s = 0$ , the only scale is set by the spin-up density, and therefore the average  $F^{(2)}$  and  $F^{(3)}$  are dimensionless constants. In general, the two- and three-quasiparticle interactions also depend on the angles between the quasiparticles close to the Fermi surface

(in particular, the effective mass at finite polaron density is given by an appropriately defined  $l = 1$  Landau parameter [45]), but for the general estimates here, we can consider an average interaction relevant for the energy contribution. If additional scales are significant, such as the effective range or other ranges  $R$ ,  $F^{(2)}$  and  $F^{(3)}$  will depend on  $R^3 n_\uparrow$ . Monte-Carlo calculations of  $F^{(2)}$  give  $F^{(2)} = 6B/5 \approx 0.17$  ( $B \approx 0.14$  in Ref. [46]), which is small (compared to the normal symmetric phase) due to the Pauli principle for spin-down fermions. These scaling results for large scattering lengths demonstrate nicely the suppression of three-quasiparticle contributions by  $(F^{(3)}/F^{(2)})(n_\downarrow/n_\uparrow)$ , in line with the results of the previous section.

### 3.3. Neutron-rich nuclei

Next we illustrate the suppression of three-quasiparticle terms for finite Fermi systems. We consider valence neutrons in neutron-rich calcium isotopes, where the interacting ground state is taken to be the  $^{40}\text{Ca}$  core.<sup>c</sup> This is also an interesting system, because recent calculations (with empirical single-particle or quasiparticle energies) have shown that the dominant contributions from chiral 3N forces are due to interactions between two valence neutrons and one core nucleon [3]. This corresponds to the normal-ordered two-body part of 3N forces, which is enhanced by the number of core nucleons. In the language of Fermi liquid theory, these are 3N force contributions to the two-quasiparticle interaction.

As shown in Fig. 1, the ground-state energies of neutron-rich calcium isotopes are well reproduced with effective two-body interactions in the shell model [47, 48]. The differences between the phenomenological interactions are however amplified with increasing neutron number.

For a finite system, the Fermi liquid expansion, Eq. (13), is given by:

$$\delta E \sim \mu \left[ N_q + \langle F^{(2)} \rangle \frac{N_q(N_q - 1)}{2A} + \langle F^{(3)} \rangle \frac{N_q(N_q - 1)(N_q - 2)}{6A^2} \right], \quad (23)$$

where we have used  $\varepsilon_1^0 \sim \mu$  and the dimensional scaling of  $f_{12}^{(2)} \sim \mu/n$  and  $f_{123}^{(3)} \sim \mu/n^2$  [see Eqs. (16) and (17)]. In this example,  $N_q$  is the number of valence neutrons and  $A = 40$  the number of core nucleons. A fit to the ground-state energies of the AME2003 atomic mass evaluation [49]

<sup>c</sup>The properties of medium-mass (and heavier) nuclei are often also calculated in energy-density functional approaches, where particle-particle (pair) correlations are included by generalizing the ground state to a (particle-number-projected) superfluid ground state. Figure 1 shows that pairing effects responsible for the odd-even-mass-staggering are relatively weak in nuclei.

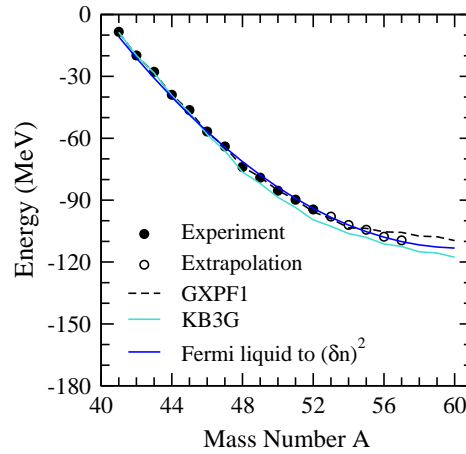


Fig. 1. Ground-state energies of calcium isotopes relative to  $^{40}\text{Ca}$  as a function of mass number  $A$ , taken from the AME2003 atomic mass evaluation [49] based on experimental energies to  $^{52}\text{Ca}$  and an extrapolation from  $^{53}\text{Ca}$  to  $^{58}\text{Ca}$  (the heaviest neutron-rich calcium isotopes known to exist [50]). In addition, we show the energies obtained from shell-model calculations with phenomenological two-body interactions KB3G [47] and GXPF1 [48], and the fit using the Fermi liquid expansion, Eq. (23), to second order.

including only two-quasiparticle interactions yields  $\mu = -10.8 \text{ MeV}$  and  $\langle F^{(2)} \rangle = -2.0$ . Overall a very good description of the energies is obtained, and the fit from  $^{41}\text{Ca}$  to  $^{58}\text{Ca}$  is not sensitive to three-quasiparticle interactions. The value for  $\mu$  is consistent with typical single-particle energies in  $^{41}\text{Ca}$  and  $\langle F^{(2)} \rangle \sim 1$  is expected for nuclear interactions. We could improve the description further by accounting for the dependence of the two-body interaction on the single-particle orbitals (as in the shell model), instead of using an average  $\langle F^{(2)} \rangle$ .

Moreover, the Fermi liquid expansion provides an estimate of the contribution from three-quasiparticle interactions. With  $\langle F^{(3)} \rangle \sim 1$ , which is likely to be an overestimate, because three-neutron interactions are suppressed by the Pauli principle, the corresponding energy contributions to  $^{52}\text{Ca}$  and  $^{58}\text{Ca}$  ( $N_q = 12$  and  $18$ ) are  $\delta E \sim 1.5$  and  $5.5 \text{ MeV}$ . This is a factor 2 smaller than the spread of the shell-model results in Fig. 1. However, due to the uncertainty in the strength of  $\langle F^{(3)} \rangle$ , microscopic calculations or more global shell-model analyses of  $\langle F^{(3)} \rangle$  are important to improve this estimate. Finally, the convergence of the Fermi liquid expansion is improved and the suppression of three-quasiparticle contributions is even

more effective in heavier nuclei.

#### 4. Concluding remarks

We have shown that for normal Fermi systems at low excitation energies the contributions of three-quasiparticle interactions are suppressed by the ratio of the quasiparticle density to the ground-state density, or equivalently by the ratio of the excitation energy over the chemical potential. This holds for excitations that conserve the number of particles (excited states of the interacting ground state) as well as for excitations that add or remove valence particles. This suppression is general and applies to strongly interacting systems even with strong, but finite-range three-body forces. However, this does not imply that the contributions from 3N forces to the interacting ground-state energy (the energy of the core nucleus in the context of shell-model calculations), to quasiparticle energies (single-particle energies), or to two-quasiparticle interactions (effective two-body interactions) are small. The argument only applies to the effects of residual three-body interactions at low energies.

The Fermi liquid suppression of three-quasiparticle interactions can be tested in large-scale shell model calculations, and with advances in ab-initio methods for larger nuclei, in no-core shell model calculations with a core [51], in coupled-cluster theory [52] and with nuclear lattice simulations [53]. For interparticle interactions where a finite-density reference state is close to the interacting ground state, the Fermi liquid expansion also implies that normal-ordered three-body interactions are small. This can explain why, for low-momentum interactions [29], calculations of nuclei [54, 55] and nucleonic matter [10, 56] at the normal-ordered two-body level are so successful.

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